

On Extensions of Superconformal Algebras

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Abstract

Starting from vector fields that preserve a differential form on a Riemann sphere with Grassmann variables, one can construct a Superconformal Algebra by considering central extensions of the algebra of vector fields. In this note, the $N=4$ case is analyzed closely, where the presence of weight zero operators in the field theory forces the introduction of non-central extensions. How this modifies the existing Field Theory, Representation Theory and Gelfand-Fuchs constructions is discussed. It is also discussed how graded Riemann sphere geometry can be used to give a geometrical description of the central charge in the $N=1$ theory.

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1 Introduction

Two dimensional conformal symmetry in Quantum Field Theory has, over the last 30 years, touched many parts of mathematics and theoretical physics. A Quantum Field Theory that is conformally invariant is called a Conformal Field Theory (CFT), and in this note, only the case of two dimensions is examined. Lying at the heart of CFT is understanding how to treat the infinite conformal symmetry on the quantum level, and understanding the representation theory of the algebra. If one wants a non-trivial, unitary representation of the symmetry algebra (known as the DeWitt algebra), then a central extension must be introduced into the algebra, yielding the Virasoro algebra[3]. Hence, when considering any algebra which has the Virasoro algebra as a subalgebra, the understanding of how and what extensions can be added is of crucial importance. Two-dimensional CFTs present an example of a Quantum Field Theory where there is a rich interplay between the geometry of the theory and the quantum theory. As a result, many aspects of the quantum theory can be described elegantly by the geometry. This is a point of view that this note will use repeatedly.

In this note, the case of the conformal symmetries of a Riemann sphere and graded Riemann sphere are examined. In [7] it was found that a graded Riemann sphere is a sensible space on which to try and construct a Superconformal Field Theory (SCFT). On this space many results can be obtained, and primary fields (fields that ‘generate’ the space of states) can be built in a natural manner associated to the geometry of the space. These spaces give rise to lie graded algebras of vector fields that contain the DeWitt algebra. Much work has been done on the extensions of these algebras in [6]. Here, the $N = 4$ case is revisited, with particular attention paid to the weight 0 fields that arise in the theory. These fields give unusual behaviour, giving logarithms in the super OPE, and exhibiting some Jordan block structure in the adjoint representation.

The structure, however, turns out to be quite manageable, since the behaviour turns out to be quite similar to that of a free boson. The understanding of how to construct bosonic primary fields geometrically, as sections of a line bundle is extremely well covered in the CFT literature. However, it is also known, in the bosonic theory, how to construct central extensions geometrically[13]. Here, this is extended to the $N = 1$ case, and is discussed how this might extend to higher N .

In Section 2, the vector fields for the $N = 4$ case are found, and extensions of the algebra considered, using just the graded Jacobi Identity. In particular, it is found that if the algebra is not reduced to its simple subalgebra, then the Jacobi identity implies that algebra must be extended by non-central elements, if it is to contain the Virasoro algebra with non-trivial central charge. Section 3 examines how this fits into the operator formalism of CFT, where the starting point is usually an Operator Product expansion. There the subtlety arises from understanding what mode expansions to take for operators, and how to treat logarithms in the Operator Product expansion. Section 4 looks at how the usual representation theory of the $N = 4$ algebra will be altered with the non-central extension found. Section 5 then considers how the algebra obtained fits into the formalism of Gelfand-Fuchs extensions, and how superspace techniques can be used to write them. Section 6 then looks at the $N = 1$ Gelfand-Fuchs cocycle, i.e. the $N = 1$ central charge, and considers how to realise the cocycle as a geometric object on a graded Riemann sphere.

2 The algebra of Vector Fields

On a Riemann sphere, one obtains the the DeWitt algebra by looking at the vector fields that preserve the one-form dz , a basis of which is given by

$$l_n = -z^{n+1} \frac{\partial}{\partial z} \quad (1)$$

Calculating the commutation relations given by the lie bracket yields the DeWitt algebra

$$[l_n, l_m] = (n - m)l_{n+m} \quad (2)$$

Similarly, on a graded Riemann sphere, with $N = 4$ and the usual one-form $\omega = dz + \sum_{i=1}^4 \theta_i d\theta_i$, one finds a basis of vector fields that preserve ω [6]

$$\begin{aligned} l_n &= -z^n \left(z\partial + \frac{1}{2}(n+1)\theta_i\partial_i \right) \\ g_r^j &= z^{r+\frac{1}{2}} \left(\theta_j\partial - \partial_j + \frac{(r+\frac{1}{2})}{z}\theta_j\theta_k\partial_k \right) \\ t_n^{ml} &= z^n \left(\theta_l\partial_m - \theta_m\partial_l + \frac{n}{z}\theta_m\theta_l\theta_p\partial_p \right) \\ \psi_r^k &= -z^{r-\frac{1}{2}} \left(\frac{1}{6}\epsilon_{kpqr}\theta_p\theta_q\theta_r\partial + \frac{(r-\frac{1}{2})}{z}\theta_1\theta_2\theta_3\theta_4\partial_k + \frac{1}{2}\epsilon_{kpqr}\theta_p\theta_q\partial_r \right) \\ u_n &= -z^{n-1} \left(\theta_1\theta_2\theta_3\theta_4\partial + \frac{1}{12}\epsilon_{ijkl}\theta_i\theta_j\theta_k\partial_l \right) \end{aligned} \quad (3)$$

These vector fields give rise to the graded commutation relations under the graded Lie bracket

$$\begin{aligned}
[l_n, l_m] &= (n - m)l_{m+n} & [l_n, g_r^j] &= (\frac{n}{2} - r)g_{n+r}^j & [l_n, t_m^{pq}] &= -mt_m^{pq} \\
[l_n, \psi_r^k] &= (-\frac{n}{2} - r)\psi_{n+r}^k & [l_n, u_m] &= -(n + m)u_{m+n} \\
[g_r^j, g_s^k] &= 2\delta_{jk}l_{r+s} + (s - r)t_{r+s}^{jk} & [t_n^{ml}, g_r^j] &= \delta_{mj}g_{n+r}^l - \delta_{lj}g_{n+r}^m + n\epsilon_{mljk}\psi_{n+r}^k \\
[g_r^j, \psi_s^k] &= \delta_{jk}2(r + s)u_{r+s} + \frac{1}{2}\epsilon_{jkpq}t_{r+s}^{pq} & [g_r^j, u_n] &= -\frac{1}{2}\psi_{n+r}^j \\
[t_n^{ml}, t_r^{pq}] &= \delta_{mq}t_{n+r}^{pl} + \delta_{mp}t_{n+r}^{lq} + \delta_{lq}t_{n+r}^{mp} + \delta_{lp}t_{n+r}^{qm} & [t_n^{ml}, u_p] &= 0 \\
[t_n^{ml}, \psi_r^k] &= \delta_{mk}\psi_{n+r}^l - \delta_{kl}\psi_{n+r}^m & [\psi_r^k, \psi_s^j] &= 0 & [\psi_r^k, u_n] &= 0 & [u_m, u_n] &= 0 \quad (4)
\end{aligned}$$

Note that this algebra is not simple, and that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}/\langle u_o \rangle$ (quotient taken in the vector space sense) is simple. This simple algebra is the large $N = 4$ algebra, without its three central elements. The t_n^{ml} form an $so(4)$ loop algebra. This loop algebra may be written explicitly as $so(4) = su(2) \oplus su(2)$ by the change of basis

$$\begin{aligned}
t_m^1 &= \frac{1}{2}(t_m^{12} + t_m^{34}) & t_m^2 &= \frac{1}{2}(t_m^{13} - t_m^{24}) & t_m^3 &= \frac{1}{2}(t_m^{14} + t_m^{23}) \\
\bar{t}_m^1 &= \frac{1}{2}(t_m^{34} - t_m^{12}) & \bar{t}_m^2 &= \frac{1}{2}(-t_m^{13} - t_m^{24}) & \bar{t}_m^3 &= \frac{1}{2}(t_m^{23} - t_m^{14})
\end{aligned}$$

These $su(2)$ s can then be centrally extended, to affine currents, with extension c and \bar{c} . Defining $c^+ = c + \bar{c}$ and $c^- = c - \bar{c}$, one finds the modified commutation relation

$$\begin{aligned}
[t_n^{ml}, t_r^{pq}] &= \delta_{mq}t_{n+r}^{pl} + \delta_{mp}t_{n+r}^{lq} + \delta_{lq}t_{n+r}^{mp} + \delta_{lp}t_{n+r}^{qm} \\
&\quad + (\delta_{mp}\delta_{lq} - \delta_{mq}\delta_{lp})c^+n\delta_{n+r,0} + \epsilon_{mlpq}c^-n\delta_{n+r,0}
\end{aligned}$$

The c^- also modifies, by the Jacobi identity, the relations

$$\begin{aligned}
[g_r^j, \psi_s^k] &= \delta_{jk}2(r + s)u_{r+s} + \frac{1}{2}\epsilon_{jkpq}t_{r+s}^{pq} + c^-(r + \frac{1}{2})\delta_{jk}\delta_{r+s,0} \\
[l_n, u_m] &= -(n + m)u_{m+n} - \frac{c^-}{4}(n + 1)\delta_{m+n,0}
\end{aligned}$$

The c^+ also modifies, by the Jacobi identity, the relations

$$\begin{aligned}
[l_m, l_n] &= (m - n)l_{m+n} - \frac{c^+}{4}m(m^2 - 1)\delta_{m+n,0} \\
[g_r^j, g_s^k] &= 2\delta_{jk}l_{r+s} + (s - r)t_{r+s}^{jk} - c^+(r^2 - \frac{1}{4})\delta_{jk}\delta_{r+s,0} \\
[\psi_r^k, \psi_s^j] &= c^+\delta_{jk}\delta_{r+s,0} & [u_m, u_n] &= -\frac{c^+}{4m}\delta_{m+n,0} \text{ for } m \neq 0
\end{aligned}$$

As it stands, the $\{g_r^j, \psi_s^j, u_0\}$ Jacobi identity implies $c^+ = 0$. This offending Jacobi identity is usually bypassed by working in $[\mathfrak{g}, \mathfrak{g}]$ rather than in \mathfrak{g} , but this is not the route that will be taken here. Non-zero c^+ can be obtained by adding another extension, denoted v_0 . From the $[l_n, u_m]$ commutator, it can be seen that the u_m form a current of weight zero. This current can be deformed to include a logarithmic term, so that $u(z) = -\sum_n u_n z^{-n} + v_0 \log z$. This then modifies the commutation relations

$$[l_n, u_m] = -(m+n)u_{m+n} - v_0 \delta_{m+n,0}$$

$$[g_r^j, \psi_s^k] = 2\delta_{jk}((r+s)u_{r+s} + v_0 \delta_{r+s,0}) + \frac{1}{2}\epsilon_{jkpq}t_{r+s}^{pq} \quad (5)$$

Using the Jacobi identity, one can see that v_0 commutes with all elements, except u_0 . The $\{g_r^j, \psi_s^j, u_0\}$ Jacobi identity now yields $[u_0, v_0] = \frac{c^+}{4}$. This algebra now realises centrally extended $K'(4)$ [6] (also known as large $N = 4$ [12]) with the u_0 operator put back in. Note that in $K'(4)$, v_0 is a central extension [6]. The behaviour of the u_m, v_0 is very similar to that of the modes of a free boson, identifying v_0 with momentum and u_0 with position. The commutation relations then become

$$[l_m, l_n] = (m-n)l_{m+n} - \frac{c^+}{4}m(m^2-1)\delta_{m+n,0} \quad [l_n, g_r^j] = (\frac{n}{2}-r)g_{n+r}^j$$

$$[l_n, \psi_r^k] = (-\frac{n}{2}-r)\psi_{n+r}^k \quad [l_n, u_m] = -(n+m)u_{m+n} - (\frac{c^-}{4}(n+1) + v_0)\delta_{m+n,0}$$

$$[g_r^j, g_s^k] = 2\delta_{jk}l_{r+s} + (s-r)t_{r+s}^{jk} - c^+(r^2 - \frac{1}{4})\delta_{jk}\delta_{r+s,0} \quad [l_n, t_m^{pq}] = -mt_m^{pq}$$

$$[t_n^{ml}, g_r^j] = \delta_{mj}g_{n+r}^l - \delta_{lj}g_{n+r}^m + n\epsilon_{mljk}\psi_{n+r}^k \quad [g_r^j, u_n] = -\frac{1}{2}\psi_{n+r}^j$$

$$[g_r^j, \psi_s^k] = \delta_{jk}2(r+s)u_{r+s} + \frac{1}{2}\epsilon_{jkpq}t_{r+s}^{pq} + (c^-(r + \frac{1}{2}) + 2v_0)\delta_{jk}\delta_{r+s,0}$$

$$[t_n^{ml}, t_r^{pq}] = \delta_{mq}t_{n+r}^{pl} + \delta_{mp}t_{n+r}^{lq} + \delta_{lq}t_{n+r}^{mp} + \delta_{lp}t_{n+r}^{qm} \\ + (\delta_{mp}\delta_{lq} - \delta_{mq}\delta_{lp})c^+n\delta_{n+r,0} + \epsilon_{mlpq}c^-n\delta_{n+r,0}$$

$$[t_n^{ml}, u_p] = 0 \quad [t_n^{ml}, \psi_r^k] = \delta_{mk}\psi_{n+r}^l - \delta_{kl}\psi_{n+r}^m \quad [\psi_r^k, \psi_s^j] = c^+\delta_{jk}\delta_{r+s,0}$$

$$[\psi_r^k, u_n] = 0 \quad [u_m, u_n] = -\frac{c^+}{4m}\delta_{m+n,0} \quad [u_0, v_0] = \frac{c^+}{4} \quad (6)$$

with c^\pm central, and v_0 has only one non-trivial commutator, namely $[u_0, v_0]$. Thus, whilst u_0, v_0 can both be considered to be operators at level 0, they cannot both be in the Cartan subalgebra. If v_0 is non-zero, one can choose it to be in the Cartan subalgebra. Usually, in a Conformal Field Theory, one finds that the space of operators at level zero can be identified with the Cartan subalgebra. This is not the case here, and can potentially lead to Jordan Blocks. In this sense, the $N = 4$ theory can be thought of as a logarithmic theory, with the logarithmic character coming from

$u(z)$. The usual large algebra comes from looking at the simple subalgebra obtained from identifying $u_0 \sim 0$, i.e. considering the field $\partial u(z)$ as fundamental rather than $u(z)$. To see where the logarithms actually come in, one must look at the operator formalism.

3 The Operator Approach

The super Operator Product Expansion of the super Virasoro operator with a primary superfield for the $N = 4$ case is given by[5]

$$\begin{aligned} \mathbb{T}(Z_1)\Phi(Z_2) \sim & \frac{h\theta_{12,1}\theta_{12,2}\theta_{12,3}\theta_{12,4}}{Z_{12}^2}\Phi(Z_2) + \frac{\theta_{12,1}\theta_{12,2}\theta_{12,3}\theta_{12,4}}{Z_{12}}\partial\Phi(Z_2) \\ & + \frac{1}{12}\frac{\epsilon_{ijkl}\theta_{12,i}\theta_{12,j}\theta_{12,k}}{Z_{12}}D_l\Phi(Z_2) + \frac{1}{4}\frac{\epsilon_{ijkl}\theta_{12,i}\theta_{12,j}J^{kl}}{Z_{12}}\Phi(Z_2) \\ & + p\log(Z_{12})\Phi(Z_2) \end{aligned} \quad (7)$$

where $Z_1 = (w, \chi_i)$, $Z_2 = (z, \theta_i)$, $\theta_{12,i} = (\chi_i - \theta_i)$, $Z_{12} = (w - z - \chi_i\theta_i)$,

$$\mathbb{T}(Z_2) = \theta_1\theta_2\theta_3\theta_4L(z) + \frac{1}{12}\epsilon_{ijkl}\theta_i\theta_j\theta_kG^l(z) + \frac{1}{8}\epsilon_{ijkl}\theta_i\theta_jT^{kl}(z) + \frac{1}{2}\theta_k\psi^k(z) - U(z) \quad (8)$$

and the J^{ab} form an $so(4)$ algebra with commutation relations given by $\frac{1}{2}t_0^{ab}$. $\log(Z_{12})$ is defined by

$$\log(Z_{12}) = \log(w - z) - \sum_{p=1}^4 \frac{1}{p} \left(\frac{\chi_i\theta_i}{(w - z)} \right)^p \quad (9)$$

The χ_i components of each side can be taken, giving

$$\begin{aligned} L(w)\Phi(Z_2) \sim & \left(\frac{h}{(w - z)^2} + \frac{1}{(w - z)}\partial + \frac{1}{2(w - z)^2}\theta_i\partial_i - \frac{\theta_i\theta_jJ^{ij}}{(w - z)^2} \right. \\ & \left. - \frac{6\theta_1\theta_2\theta_3\theta_4}{(w - z)^4}p \right)\Phi(Z_2) \\ \frac{1}{2}G^i(w)\Phi(Z_2) \sim & \left(\frac{-h\theta_i}{(w - z)^2} - \frac{\theta_i\partial}{2(w - z)} + \frac{\partial_i}{2(w - z)} - \frac{\theta_i\theta_j\partial_j}{2(w - z)^2} \right. \\ & \left. + \frac{\theta_i\theta_j\theta_kJ^{jk}}{(w - z)^3} + \frac{\theta_jJ^{ji}}{(w - z)^2} - \frac{\epsilon_{ijkl}\theta_j\theta_k\theta_l}{3(w - z)^3}p \right)\Phi(Z_2) \\ \frac{1}{2}T^{ab}(w)\Phi(Z_2) \sim & \left(\frac{h\theta_a\theta_b}{(w - z)^2} + \frac{\theta_b\partial_a - \theta_a\partial_b}{2(w - z)} + \frac{\theta_a\theta_b\theta_j\partial_j}{2(w - z)^2} - \frac{\theta_1\theta_2\theta_3\theta_4}{(w - z)^3}\epsilon_{abjk}J^{jk} \right. \\ & + \frac{1}{(w - z)^2}(\theta_a\theta_jJ^{bj} - \theta_b\theta_jJ^{aj}) + \frac{1}{(w - z)}J^{ab} \\ & \left. + \frac{\epsilon_{abjk}\theta_j\theta_k}{2(w - z)^2}p \right)\Phi(Z_2) \end{aligned}$$

$$\begin{aligned}
\frac{1}{2}\psi^k(w)\Phi(Z_2) &\sim \left(\frac{-h\epsilon_{klmn}\theta_l\theta_m\theta_n}{6(w-z)^2} + \frac{\epsilon_{klmn}\theta_l\theta_m\theta_n}{12(w-z)}\partial + \frac{\theta_1\theta_2\theta_3\theta_4}{2(w-z)^2}\partial_k + \frac{\epsilon_{klmn}\theta_l\theta_m}{4(w-z)}\partial_n \right. \\
&\quad \left. - \frac{\epsilon_{klmn}\theta_l}{2(w-z)}J^{mn} + \frac{\theta_k\epsilon_{lmnp}\theta_l\theta_m}{4(w-z)^2}J^{np} - \frac{\theta_k}{(w-z)}p \right)\Phi(Z_2) \\
-U(w)\Phi(Z_2) &\sim \left(\frac{h\theta_1\theta_2\theta_3\theta_4}{(w-z)^2} - \frac{\theta_1\theta_2\theta_3\theta_4}{(w-z)}\partial - \frac{\epsilon_{klmn}\theta_k\theta_l\theta_m}{12(w-z)}\partial_n + \frac{\epsilon_{klmn}\theta_k\theta_l}{4(w-z)}J^{mn} \right. \\
&\quad \left. + \log(w-z)p \right)\Phi(Z_2)
\end{aligned} \tag{10}$$

from which the vector fields of (3) can be recovered. Note that logarithms only appear in OPEs containing $U(z)$. Clearly, in this last OPE, a contour integral can only be taken if $\partial U(w)\Phi(z)$ is considered. Taking $U(z) = \sum_n U_n z^{-n} + V_0 \log z$, it can be shown that $[V_0, \Phi] = p\Phi$. Allowing V_0 to annihilate the vacuum [14] then yields $V_0|\Phi\rangle = p|\Phi\rangle$. The $[V_0, \Phi]$ commutator is unusual, in that it contains no differential operators. Hence, it is not obvious how to associate a conformal vector field of the form of (3) to V_0 . The logarithm in the last OPE of (10) prevents one from obtaining an action from U_0 . If one looks at a representation where $V_0 = 0$, then it can be seen that $p = 0$ and that the w contour integral in $U(w)\Phi(z)$ can be performed, to give.

$$[U_0, \Phi(Z)] \sim \left(\frac{h\theta_1\theta_2\theta_3\theta_4}{z^2} + \frac{\theta_1\theta_2\theta_3\theta_4\partial}{z} + \frac{\epsilon_{klmn}\theta_k\theta_l\theta_m\partial_n}{12z} - \frac{\epsilon_{klmn}\theta_k\theta_l J^{mn}}{4z} \right)\Phi(Z)$$

However, its action on the highest weight from this approach is unclear, and a more careful approach to the representation theory is warranted.

The logarithmic character can also be examined by looking at the $\mathbb{T}(Z_1)\mathbb{T}(Z_2)$ OPE, given by

$$\begin{aligned}
\mathbb{T}(Z_1)\mathbb{T}(Z_2) &\sim \frac{c^+ \log(Z_{12})}{4} - \frac{c^- \theta_{12,1}\theta_{12,2}\theta_{12,3}\theta_{12,4}}{4Z_{12}^2} + \frac{\theta_{12,1}\theta_{12,2}\theta_{12,3}\theta_{12,4}}{Z_{12}}\partial\mathbb{T}(Z_2) + \\
&\quad \frac{1}{12} \frac{\epsilon_{ijkl}\theta_{12,i}\theta_{12,j}\theta_{12,k}}{Z_{12}} D_l \mathbb{T}(Z_2)
\end{aligned} \tag{11}$$

Using (9), one can see that the only term involving logarithms is the term

$$U(w)U(z) \sim \frac{c^+}{4} \log(w-z) \tag{12}$$

Therefore, if $c^+ \neq 0$, $U(z)$ must have a logarithmic component in its mode expansion. This can easily be verified by taking the contour integrals in (12) to get the commutation relations. If there is no logarithmic component in $U(z)$, then from the computation one can deduce that $c^+ = 0$, the same result found when using the Jacobi identity in the previous section. The field $U(z)$ behaves in a very similar way to a free boson. In particular, $[V_0, U_0] = \frac{c^+}{4}$, and hence they are not mutually diagonalizable, as was reflected by the above manipulations of the last OPE in (10).

One can ask if the field, $U(z)$, can be written in a logarithmic form [1]

$$\begin{aligned} L(w)D(z) &\sim \frac{hD(z) + E(z)}{(z-w)^2} + \frac{\partial D(z)}{(w-z)} \\ L(w)E(z) &\sim \frac{hE(z)}{(z-w)^2} + \frac{\partial E(z)}{(w-z)} \end{aligned}$$

From (11) one can read off the $L(w)U(z)$ OPE to find

$$L(w)U(z) \sim \frac{\partial U(z)}{(w-z)} - \frac{c^-}{4(w-z)^2} \quad (13)$$

Regarding $-\frac{c^-}{4}$ as a constant field of weight zero, one then gets the desired form. Whilst the V_0 operator gives rise to an eigenvalue, this analysis does not yield a conformal transformation associated to V_0 . The previous analysis shows that v_0 had to be introduced as an extension of the algebra. This suggests that rather than thinking of the p eigenvalue as being associated to a primary field, one should instead think of the V_0 operator as appearing in a similar way to a central extension.

4 A little Representation Theory

A closer look at the level zero operators is warranted. These are normally defined as those operators with $\text{ad}l_0$ eigenvalue being zero. Considering the commutation relations (6), one can see that clearly $\{l_0, t_0^{ml}, v_0, c^\pm\}$ fall into this category. However, u_0 has a strange action under $\text{ad}l_0$, namely

$$[l_0, u_0] = -\left(\frac{c^-}{4} + v_0\right)$$

If one were to define a basis $e_1 = -\frac{c^-}{4} - v_0$, $e_2 = u_0$ and write down the matrix for $\text{ad}l_0$ with respect to this basis, one would find the Jordan block

$$(\text{ad}l_0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

In this manner, u_0 can be considered as an operator at level zero that is not in the Cartan subalgebra.

The algebra here differs slightly from the usual large $N = 4$ algebra, by the mode u_0 . This affects the representation theory[10][11]. The Fock space will be enlarged, due to the presence of polynomials in u_0 acting on the highest weight. Using the analogy of v_0 as momentum, and u_0 as position, instead of considering the states $u_0^n|h\rangle$, the ‘momentum’ eigenstates $|k, h\rangle = e^{-ku_0}|h\rangle$ can be considered. From the fact that the only non-trivial commutators u_0 has are with l_n, g_r^j, v_0 , one can show

$$\begin{aligned} v_0|k, h\rangle &= \left(\frac{c^+k}{4} + p\right)|k, h\rangle, & g_r^j|k, h\rangle &= e^{-ku_0}g_r^j|h\rangle + \frac{k}{2}\psi_r^j|k, h\rangle \\ l_n|k, h\rangle &= \begin{cases} e^{-ku_0}l_n|h\rangle + knu_n|k, h\rangle & n \neq 0 \\ \left(h + k\left(\frac{c^-}{4} + p\right) + k^2\frac{c^+}{8}\right)|k, h\rangle & n = 0 \end{cases} \end{aligned} \quad (14)$$

from which it can be seen that $|k, h\rangle$ obeys highest weight conditions, with potentially different v_0 and l_0 eigenvalues from $|h\rangle$. In analogy with a free boson, the u_n , $n > 0$ annihilate the vacuum and the highest weight state and v_0 annihilates the vacuum. For non-zero c^+ , u_0 annihilates neither.

5 Gelfand-Fuchs 2-cocycles

For an algebra of vector fields, where a function can be associated to each vector field, it is often useful to construct central extensions by considering Gelfand-Fuchs 2-cocycles[4]. Here, superfield formalisms are used, which yield similar results to [4]. For instance, in the bosonic case, one has[13]

$$l_m = -z^{m+1} \frac{\partial}{\partial z}, \quad l(z) = z^{m+1}, \text{ then} \\ c(l_m, l_n) = \frac{1}{24\pi i} \oint_0 dz \left(\frac{\partial^3}{\partial z^3} z^{m+1} \right) z^{n+1} = \frac{c}{12} m(m^2 - 1) \delta_{m+n,0} \quad (15)$$

or for a general polynomial $l(z)$

$$c(l_{(1)}, l_{(2)}) = \frac{1}{24\pi i} \oint_0 dz l'''_{(1)} l_{(2)} \quad (16)$$

where the contour is a closed loop around the origin, say the unit circle, beginning and ending at $z = 1$. Since the extensions are known for the $N = 1, 2, 3, 4$ algebras, they can be put in Gelfand-Fuchs 2-cocycle form. For the $N = 1$ case

$$l_n = -z^n (z \partial_z + \frac{1}{2}(n+1) \theta \partial_\theta), \quad l(z) = z^{n+1} \\ g_r = z^{r+\frac{1}{2}} (\partial_\theta - \theta \partial_z), \quad g(z) = z^{r+\frac{1}{2}} \quad (17)$$

Defining the graded field $X_{(i)} = \frac{1}{2} l_{(i)} + \theta g_{(i)}$ with l and g graded even polynomials in z, z^{-1} , one finds the central extension is given by the 2-cocycle

$$c(X_{(1)}, X_{(2)}) = \frac{1}{6\pi i} \oint_0 dz d\theta (D X''_{(1)}) X_{(2)} \quad (18)$$

where $D = \partial_\theta + \theta \partial_z$, and, as usual, $\int d\theta$ really means $\frac{\partial}{\partial \theta}$. Similarly, for $N = 2$, the vector fields and associated fields are

$$l_n = -z^n (z \partial_z + \frac{1}{2}(n+1) \theta_i \partial_{\theta_i}), \quad l(z) = z^{n+1} \\ g_r^i = z^{r-\frac{1}{2}} (z \theta_i \partial_z - z \partial_{\theta_i} + (r + \frac{1}{2}) \theta_i \theta_j \partial_{\theta_j}), \quad g^i(z) = z^{r+\frac{1}{2}} \\ t_m = -z^m (\theta_2 \partial_{\theta_1} - \theta_1 \partial_{\theta_2}), \quad t(z) = z^m \quad (19)$$

Introducing the graded field

$$X_{(i)} = \frac{1}{2} l_{(i)} + \theta_j g_{(i)}^j + \theta_1 \theta_2 t_{(i)} \quad (20)$$

the 2-cocycle is given by

$$c(X_{(1)}, X_{(2)}) = \frac{1}{6\pi i} \oint_0 dz d\theta_2 d\theta_1 (D_1 D_2 X'_{(1)}) X_{(2)} \quad (21)$$

where $D_i = \partial_{\theta_i} + \theta_i \partial_z$. For $N = 3$,

$$\begin{aligned} l_n &= -z^n(z\partial_z + \frac{1}{2}(n+1)\theta_i\partial_{\theta_i}), & l(z) &= z^{m+1} \\ g_r^i &= z^{r-\frac{1}{2}}(z\theta_i\partial_z - z\partial_{\theta_i} + (r+\frac{1}{2})\theta_i\theta_j\partial_{\theta_j}), & g^i(z) &= z^{r+\frac{1}{2}} \\ t_m^i &= z^{m-1}(z\epsilon_{ijk}\theta_j\partial_{\theta_k} - m\theta_1\theta_2\theta_3\partial_{\theta_i}), & t^i(z) &= z^m \\ \psi_r &= -z^{r-\frac{1}{2}}(\theta_1\theta_2\theta_3\partial_z + \frac{1}{2}\epsilon_{ijk}\theta_i\theta_k\partial_{\theta_j}), & \psi(z) &= z^{r-\frac{1}{2}} \end{aligned} \quad (22)$$

$$X_{(i)} = \frac{1}{2}l_{(i)} + \theta_j g_{(i)}^j + \frac{1}{2}\epsilon_{klm}\theta_k\theta_l t_{(i)}^m + \theta_1\theta_2\theta_3\psi_{(i)} \quad (23)$$

$$c(X_{(1)}, X_{(2)}) = \frac{1}{6\pi i} \oint_0 dz d\theta_3 d\theta_2 d\theta_1 (D_1 D_2 D_3 X_{(1)}) X_{(2)} \quad (24)$$

Now, for $N = 4$, $X_{(i)}$ is given by

$$X_{(i)} = \frac{1}{2}l_{(i)} + \theta_j g_{(i)}^j + \frac{1}{2}\theta_a\theta_b t_{(i)}^{ab} - \frac{1}{6}\epsilon_{abcd}\theta_a\theta_b\theta_c\psi_{(i)}^d - \theta_1\theta_2\theta_3\theta_4\frac{1}{2}u_{(i)} \quad (25)$$

where $u_{(i)} = z^{n-1}$ corresponds to the vector u_m in (3), and similarly for the other fields in $X_{(i)}$. In the cases so far, given an $X_{(i)}$, a conformal vector field can be obtained. It is worth considering what $X_{(i)}$ means in the operator approach. To this end, recall the super stress-energy tensor \mathbb{T} from (8). From the OPE (11), it can be shown that $L(z)$ scales like a field of weight 2, and hence expansion $L(z) = \sum_m L_m z^{-m-2}$. Similarly, the other operators have scaling dimensions - G^i is $\frac{3}{2}$, T^{ij} is 1, ψ^i is $\frac{1}{2}$ and U is 0. In fact, the G^i, T^{ij}, ψ^i are primary fields. Now, rather than obtain the vector field associated to $X_{(i)}$, the operator associated to it can be obtained by computing

$$\frac{1}{\pi i} \oint_0 dz d\theta_4 d\theta_3 d\theta_2 d\theta_1 X_{(i)} \mathbb{T} \quad (26)$$

A similar formula [15] holds for the smaller N . Since V_0 is a part of \mathbb{T} , one can ask how to obtain the operator V_0 from the above integral, and see if it sheds light on how the $[u_0, v_0] = \frac{c^+}{4}$ commutator might be obtained. To this end, consider the logarithmic part of

$$\begin{aligned} \frac{1}{\pi i} \int_{1+i\epsilon}^{1-i\epsilon} dz \frac{1}{2} u(z) U(z) &= \frac{1}{\pi i} \int_{1+i\epsilon}^{1-i\epsilon} dz \frac{1}{2} u(z) V_0 \log(z) \\ &= -\frac{1}{2\pi i} \int_{1+i\epsilon}^{1-i\epsilon} dz \frac{1}{z} V_0 \int u + \frac{1}{2\pi i} V_0 [\log(z) \int u]_{1+i\epsilon}^{1-i\epsilon} \end{aligned} \quad (27)$$

Concentrating on the first part of the expression, it seems very suggestive to associate the constant part of $\int u$, which arises as an integration constant, to the algebra element v_0 (assuming $\int u$ is single valued around the origin, i.e. u has no $\frac{1}{z}$ term). This turns out to be precisely what is needed to obtain the c^+ 2-cocycle.

$$c^+(X_{(1)}, X_{(2)}) = \frac{-1}{2\pi i} \oint_0 dz d\theta_4 d\theta_3 d\theta_2 d\theta_1 (D_1 D_2 D_3 D_4 \int X_{(1)}) X_{(2)} \quad (28)$$

assuming the integrand is single valued around the origin. On expanding out the $X_{(i)}$ and applying all the superderivatives, the only component of $X_{(1)}$ that is actually integrated in $D_1 D_2 D_3 D_4 \int X_{(1)}$ is the $u_{(1)}$ component. This essentially means that $c^+(u_0, u_0)$ cannot be explicitly obtained, since the integrand would have logs in it. This, however, is not a problem, since $[u_0, u_0] = 0$ by antisymmetry of the commutator. Also, $c^+(v_0, v_0)$ is not obtained, but by the same argument is clearly zero. Most importantly, if the integration constant is taken to be 1, then one can obtain $c^+(v_0, u_0) = -\frac{1}{4}$. The c^- cocycle can also be found,

$$c^-(X_{(1)}, X_{(2)}) = -\frac{1}{2\pi i} \oint_0 dz d\theta_4 d\theta_3 d\theta_2 d\theta_1 X'_{(1)} X_{(2)} \quad (29)$$

as well as an expression for the v_0 extension.

$$v_0(X_{(1)}, X_{(2)}) = \frac{2}{\pi i} \oint_0 dz d\theta_4 d\theta_3 d\theta_2 d\theta_1 \left(\frac{1}{z} \left(1 - \frac{1}{2} \theta_i \partial_{\theta_i} \right) X_{(1)} \right) X_{(2)} \quad (30)$$

All of the extensions here for all N are consistent with the operator formalism. Apart from c^+ in $N = 4$, which has a problem with logs, all the extensions can be obtained from the super OPE by calculating

$$-\frac{1}{4\pi} \oint_0 dz d\theta_N \dots d\theta_1 \oint_z dw d\chi_N \dots d\chi_1 X_{(1)}(Z_1) X_{(2)}(Z_2) \mathbb{T}(Z_1) \mathbb{T}(Z_2) \quad (31)$$

and give rise to the same formulae. The formulae also suggest that the extensions should be described by a map from two vector fields into something proportional to the ‘volume form’

$$C : \mathcal{D}^1 \mathcal{A}_N \times \mathcal{D}^1 \mathcal{A}_N \rightarrow dz \otimes_i \frac{\partial}{\partial \theta_i} \quad (32)$$

6 $N = 1$ Gelfand-Fuchs 2-cocycle

In (17), a parameterization of a vector field in terms of a field was written. One can try and explore how these fields are related to a graded Riemann sphere. First, redefine $X = \frac{1}{2}f(z) + g(z)\theta$, where now f and g need not have a defined parity, i.e. they are each a sum of an even part and an odd part. Now, introduce the map

$$l : X \mapsto -\left(f(z) \frac{\partial}{\partial z} + \frac{1}{2} f'(z) \theta D \right) - g(z) \left(2\theta \frac{\partial}{\partial z} - D \right) \quad (33)$$

Under this identification, (18) holds. In components (18) now reads

$$c(X_{(1)}, X_{(2)}) = \frac{1}{6\pi i} \oint dz \left(\frac{1}{4} f'''_{(1)} f_{(2)} + (-1)^{g_{(2)}} g''_{(1)} g_{(2)} \right) \quad (34)$$

where [16] $(-1)^{g_{(2)}} g_{(2)} = (-1)^{g_{(2)0}} g_{(2)0} + (-1)^{g_{(2)1}} g_{(2)1} = g_{(2)0} - g_{(2)1}$ with $g_{(2)0}$ and $g_{(2)1}$ being respectively the even and odd parts of $g_{(2)}$. Recall that any invertible superconformal transformation can be parameterized by $\Phi : (z, \theta) \mapsto (w, \pi)$

$$\begin{aligned} w &= w_0 + \theta w_1 (w'_0)^{\frac{1}{2}} \\ \chi &= w_1 + \theta (w'_0 + w_1 w'_1)^{\frac{1}{2}} \end{aligned} \quad (35)$$

where $w_0 = w_0(z)$ is even and with body, and $w_1 = w_1(z)$ is odd. Restricting to invertible transformations, and using the superconformal condition $Dw = \chi D\chi$, one can show that

$$\Phi^* \begin{pmatrix} \frac{\partial}{\partial w} \\ D_\chi \end{pmatrix} = \begin{pmatrix} (D_\theta \chi)^{-2} & -\left(\frac{\partial \chi}{\partial z}\right)(D_\theta \chi)^{-3} \\ 0 & (D_\theta \chi)^{-1} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial z} \\ D_\theta \end{pmatrix} \quad (36)$$

Plugging in the parameterization (35), and looking at the pull-pack of the vector field $l(X)$, one finds the induced transformation law for X

$$X(w, \chi) \mapsto (D_\theta \chi)^{-2} (X \circ \Phi)(z, \theta) =: \hat{X}(z, \theta) \quad (37)$$

showing that X can in fact be identified with components of sections of the locally rank one sheaf ω^{-1} [7][9], where $\omega = dz + \theta d\theta$. Thus, in local co-ordinates, it reads as the ‘tensor’ $\mathbf{X} = X\omega^{-1}$, yielding the transformation law

$$\Phi^*(X(w, \chi)\omega^{-1}) = (X \circ \Phi)(z, \theta)\omega^{-1}(D_\theta \chi)^{-2} = \hat{X}(z, \theta)\omega^{-1} \quad (38)$$

A bracket can be introduced on $\Gamma(\omega^{-1})$ [6] as

$$[X_{(1)}, X_{(2)}] = -2X_{(1)}X'_{(2)} + 2X'_{(1)}X_{(2)} - (-1)^{X_{(1)}}(DX_{(1)})(DX_{(2)}) \quad (39)$$

The bracket is graded antisymmetric and obeys the graded Jacobi identity, and hence defines a lie graded algebra structure. The map $l : \Gamma(\omega^{-1}) \rightarrow \Gamma(\mathcal{D}^1 \mathcal{A}_1)$ is then a lie graded algebra homomorphism, with the bracket on vector fields given by the usual lie bracket. (18) can be rewritten as

$$c(X_{(1)}, X_{(2)}) = \frac{1}{12\pi i} \oint_0 dz d\theta \left((DX''_{(1)})X_{(2)} - (-1)^{X_{(1)}X_{(2)}}(DX''_{(2)})X_{(1)} \right) \quad (40)$$

The integration ‘measure’, transforms, according to the Berezinian as

$$\Phi^* \left[dw \frac{\partial}{\partial \chi} \right] = \left[dz \frac{\partial}{\partial \theta} \right] (D_\theta \chi) \quad (41)$$

Knowing how X transforms from (37), one can compute

$$\begin{aligned} \Phi^* \left((D_\chi X''_{(1)})X_{(2)} - (-1)^{X_{(1)}X_{(2)}}(D_\chi X''_{(2)})X_{(1)} \right) = \\ (D_\theta \chi)^{-1} \left((D_\theta \hat{X}''_{(1)})\hat{X}_{(2)} - (-1)^{X_{(1)}X_{(2)}}(D_\theta \hat{X}''_{(2)})\hat{X}_{(1)} + 2\{\chi, \theta\}[\hat{X}_{(1)}, \hat{X}_{(2)}] \right. \\ \left. + D_\theta(\{\chi, \theta\})((D_\theta \hat{X}_{(1)})\hat{X}_{(2)} - (-1)^{X_{(1)}X_{(2)}}(D_\theta \hat{X}_{(2)})\hat{X}_{(1)}) \right) \end{aligned} \quad (42)$$

where the primes on the left hand side are derivatives with respect to w , and on the right hand side with respect to z . Also, $(-1)^X \hat{X} = (-1)^{\hat{X}} \hat{X}$ on virtue of $D_\theta \chi$ being even and

$$\{\chi, \theta\} = \frac{\chi''}{D_\theta \chi} - 2 \frac{\chi' D_\theta \chi'}{(D_\theta \chi)^2} \quad (43)$$

is the $N = 1$ Schwarzian[8]. Notice that the last term in (42) is a total derivative in D_θ , and hence will vanish under the integral. Hence, it is most useful to look at sections of $dz \otimes \frac{\partial}{\partial \theta}$ modulo exact derivatives. More explicitly,

$$\oint_0 dz d\theta (f_0 + f_1 \theta) = (-1)^{f_1} \oint_0 dz f_1 \quad (44)$$

and modulo exact derivatives means that that if f_1 has an antiderivative, then $(f_0 + f_1 \theta) \sim 0$. In particular, this means

$$f_0 + f_1 \theta = f_0 + F_1' \theta = D_\theta((-1)^{F_1} F_1 + (-1)^{f_0} f_0 \theta) \sim 0 \quad (45)$$

as required.

Given $\Phi : (z, \theta) \mapsto (w, \chi)$, a contravariant map can be defined

$$U_{\chi, \theta} : \left(\begin{array}{c} \frac{\epsilon}{6}((D_\chi X''_{(1)})X_{(2)} - (-1)^{X_{(1)}X_{(2)}}(D_\chi X''_{(2)})X_{(1)}) \\ [X_{(1)}, X_{(2)}] \end{array} \right) \mapsto \left(\begin{array}{c} \frac{\epsilon}{6}((D_\theta \hat{X}''_{(1)})\hat{X}_{(2)} - (-1)^{X_{(1)}X_{(2)}}(D_\theta \hat{X}''_{(2)})\hat{X}_{(1)}) \\ [\hat{X}_{(1)}, \hat{X}_{(2)}] \end{array} \right) \quad (46)$$

by

$$U_{\chi, \theta} \left(\begin{array}{c} \frac{\epsilon}{6}((D_\chi X''_{(1)})X_{(2)} - (-1)^{X_{(1)}X_{(2)}}(D_\chi X''_{(2)})X_{(1)}) \\ [X_{(1)}, X_{(2)}] \end{array} \right) = \left(\begin{array}{cc} (D_\theta \chi) & -\frac{\epsilon}{3}\{\chi, \theta\}(D_\theta \chi)^{-2} \\ 0 & (D_\theta \chi)^{-2} \end{array} \right) \Phi^* \left(\begin{array}{c} \frac{\epsilon}{6}((D_\chi X''_{(1)})X_{(2)} - (-1)^{X_{(1)}X_{(2)}}(D_\chi X''_{(2)})X_{(1)}) \\ [X_{(1)}, X_{(2)}] \end{array} \right) \quad (47)$$

recalling that the first term is only defined up to exact derivatives. Given, in addition, a map $\Psi : (w, \chi) \mapsto (u, \rho)$, and using the property of the Schwarzian $\{\rho, \theta\} = \{\rho, \chi\}(D_\theta \chi)^3 + \{\chi, \theta\}$ which can be deduced from (43), one can show

$$U_{\chi, \theta} \circ U_{\rho, \chi} = U_{\rho, \theta} \quad (48)$$

Associating the necessary open sets to the maps Φ and Ψ then realises (48) as the requirement on restriction maps. This then, locally, represents a nontrivial extension of

$$\left[dz \frac{\partial}{\partial \theta} \right] \oplus \omega^{-1} \quad (49)$$

modulo exact derivatives in the first slot and for $c \neq 0$. If $c = 0$, the extension becomes trivial. A graded lie bracket can then be defined on sections of this extension as

$$\left[\left(\begin{array}{c} \lambda \\ X_{(1)} \end{array} \right), \left(\begin{array}{c} \mu \\ X_{(2)} \end{array} \right) \right] = \left(\begin{array}{c} \frac{\epsilon}{6}((D_\chi X''_{(1)})X_{(2)} - (-1)^{X_{(1)}X_{(2)}}(D_\chi X''_{(2)})X_{(1)}) \\ [X_{(1)}, X_{(2)}] \end{array} \right) \quad (50)$$

where the grade is given by the lower component. Note that, although it looks like the top component has a different parity to the bottom component, after performing

the $\int dzd\theta$ integral to get the central charge, it will have the same parity. The graded Jacobi identity is already satisfied by the lower component, it remains to verify the upper component.

$$(-1)^{X_{(1)}X_{(3)}} \left[\left[\begin{pmatrix} \lambda \\ X_{(1)} \end{pmatrix}, \begin{pmatrix} \mu \\ X_{(2)} \end{pmatrix} \right], \begin{pmatrix} \sigma \\ X_{(3)} \end{pmatrix} \right] + \text{cyclic} = \begin{pmatrix} U \\ 0 \end{pmatrix} \quad (51)$$

has top component

$$\begin{aligned} U &= (-1)^{X_{(1)}X_{(3)}} \left(\left(- (DX_{(1)})X_{(2)}''' + (-1)^{X_{(1)}}X_{(1)}'''(DX_{(2)}) - 3(-1)^{X_{(1)}}X_{(1)}'(DX_{(2)}'') \right. \right. \\ &\quad \left. \left. + 3(DX_{(1)}'')X_{(2)}' - 2(-1)^{X_{(1)}}X_{(1)}(DX_{(2)}''') + 2(DX_{(1)}''')X_{(2)} \right) X_3 + (-1)^{X_{(1)}+X_{(2)}} \left(2X_{(1)}X_{(2)}' \right. \right. \\ &\quad \left. \left. - 2X_{(1)}'X_{(2)} + (-1)^{X_{(1)}}(DX_{(1)}DX_{(2)}) \right) \right) (DX_{(3)}'') + \text{cyclic} \\ &= (-1)^{X_{(1)}X_{(3)}} \partial \left(2(DX_{(1)}')X_{(2)}'X_{(3)} - 2(DX_{(1)}')X_{(2)}X_{(3)}' + 2(-1)^{X_{(1)}}X_{(1)}(DX_{(2)}')X_{(3)}' \right. \\ &\quad \left. - 2(-1)^{X_{(1)}}X_{(1)}(DX_{(2)}')X_{(3)} + 2(-1)^{X_{(1)}+X_{(2)}}X_{(1)}'X_{(2)}(DX_{(3)}') - \right. \\ &\quad \left. 2(-1)^{X_{(1)}+X_{(2)}}X_{(1)}X_{(2)}'(DX_{(3)}') - (-1)^{X_{(2)}}\partial((DX_{(1)})(DX_{(2)})(DX_{(3)})) \right) + \\ &\quad (-1)^{X_{(1)}X_{(3)}} D \left(2(-1)^{X_{(1)}}(DX_{(1)}') - 2(-1)^{X_{(1)}+X_{(2)}}(DX_{(1)}')X_{(2)}(DX_{(3)}') + \right. \\ &\quad \left. 2(-1)^{X_{(2)}}X_{(1)}(DX_{(2)}')(DX_{(3)}') + (-1)^{X_{(1)}}(DX_{(1)}'')(DX_{(2)})X_{(3)} - \right. \\ &\quad \left. (-1)^{X_{(1)}+X_{(2)}}(DX_{(1)}'')X_{(2)}(DX_{(3)}) + (-1)^{X_{(1)}}(DX_{(1)})(DX_{(2)}'')X_{(3)} + \right. \\ &\quad \left. (-1)^{X_{(2)}}X_{(1)}(DX_{(2)}'')(DX_{(3)}) + (-1)^{X_{(2)}}X_{(1)}(DX_{(2)})(DX_{(3)}'') - \right. \\ &\quad \left. (-1)^{X_{(1)}+X_{(2)}}(DX_{(1)})X_{(2)}(DX_{(3)}'') \right) \sim 0 \end{aligned} \quad (52)$$

where the equivalence follows since the term is a total derivative (recall $\partial = D^2$), and the bracket obeys the graded Jacobi identity. Hence, the algebra of conformal vector fields admits an extension on the graded Riemann sphere to accommodate the central charge, which realises a map of the form (32) using (18). The construction precisely mirrors that done for the bosonic case in [13]. Given the list of Schwarzians in [8], it should be possible to extend the construction to all $N \leq 4$.

7 Conclusions

If one wishes to study all of the conformal symmetries on an $N = 4$ graded Riemann sphere at the quantum level, and not neglect any symmetries, then one is forced to look at adding non-central extensions to the algebra. If the u_0 symmetry is neglected, then all of the extensions are central. To the author's knowledge, this has not been observed before, and the author is unaware of the $N = 4$ algebra given by (6) having appeared previously in the literature. It was found that this extension was completely consistent with the usual CFT treatments of the quantum theory, i.e. with the Operator Product Expansions of the $N = 4$ theory, and in fact explained why the logarithms appeared in the OPE. The effect on the representation theory was discussed, as was the description

of the Gelfand-Fuchs extensions for $N = 1 \dots 4$, given by the superfield formalism. There, it was found that the extra extension explained the appearance of the indefinite integral inside the contour integral in (28). It was then described for the $N = 1$ case how the central extension arises from the geometry of an $N = 1$ graded Riemann sphere, and how it might be expected that the central extensions for the higher N might be obtained.

As well as exploiting graded Riemann sphere geometry to describe extensions of Superconformal algebras, these calculations also shed some light on how one might try to treat non-central extensions in a CFT. In this case, the Field Theory exhibited an, albeit mild, logarithmic behaviour, not unrelated to the manner in which zero modes are modified in [2]. For higher N , the requirement of having a Virasoro subalgebra with non-zero central extension could force non-central extensions to appear that enrich the zero mode structure. $N = 5$ would have a graded odd weight $-\frac{1}{2}$ field whose zero mode structure might give unavoidable Jordan cells in the representation theory. In the $N = 4$ case, the modified zero mode structure was reflected by the appearance of indefinite integrals in the Gelfand-Fuchs extensions.

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- [14] if $U(z)|0\rangle$ is to be regular at $z = 0$, then V_0 must annihilate $|0\rangle$
- [15] e.g. in the $N = 0$ case $\frac{1}{\pi i} \oint_0 dz X_{(i)} \mathbb{T} = \frac{1}{2\pi i} \oint_0 dz l_{(i)} L(z)$. The vector l_n is parameterized by $l_{(i)} = z^{n+1}$. Then $\frac{1}{2\pi i} \oint_0 dz l_{(i)} L(z) = \frac{1}{2\pi i} \oint_0 z^{n+1} \sum_m L_m z^{-m-2} = L_n$
- [16] more generally, $(-1)^{\Pi_i X_{(i)} \Pi_i X_{(i)}} = \sum_{n_i \in \{0,1\}} (-1)^{\Pi_i X_{(i)} n_i} \Pi_i X_{(i) n_i}$